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Scaled asymptotics for some q-functions

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ABSTRACT

In this work we investigate Plancherel–Rotach type asymptotics for some q-series as $q \to 1$ in a more general setting by introducing admissible sequences. These q-series generalize Ramanujan function $A_q(z)$ (a.k.a. q-Airy function), Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, q^{-1}-Hermite polynomials $h_n(x|q)$, Stieltjes–Wigert polynomials $S_n(x;q)$, q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series.

1. Introduction

The asymptotics of q-special functions is not only interesting mathematically, it also has many applications in physics. Several authors investigated the fixed q asymptotics for certain q-special functions by using various methods, for example see [2,3,5]. In [7,8] we found a method and applied it systematically to Plancherel–Rotach type asymptotics for three sets of orthogonal polynomials related to indeterminate moment problems. This method was later extended in [13–15] to study certain Plancherel–Rotach type asymptotics for some q-series including Ramanujan's entire function $A_q(z)$,

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Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, q^{-1}-Hermite polynomials $h_n(x|q)$, Stieltjes–Wigert polynomials $S_n(x;q)$, q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series, we were able to get more precise remainders. In this work we employ these asymptotic formulas from [13] to study the scaled asymptotics of these q-series as $q \to 1$. By using admissible sequences we are able to vastly generalize the results in [15].

In Section 2 we list some notations. We present our results in Section 3 and their proofs in Section 4. Throughout this work we always assume that $0 < q < 1$ unless otherwise stated.

2. Preliminaries

For $z \in \mathbb{C}$, as in [1,4,6,9], we define $(z;q)_\infty$ and $\Gamma_q(z)$ by

$$(z;q)_\infty = \prod_{k=0}^{\infty}\bigl(1 - zq^k\bigr), \qquad \frac{1}{\Gamma_q(z)} = \frac{(q^z;q)_\infty (1-q)^{z-1}}{(q;q)_\infty}. \tag{2.1}$$

For each fixed $z \in \mathbb{C}$, $\Gamma_q(z)$ is related to $\Gamma(z)$ via $\lim_{q\uparrow 1} \frac{1}{\Gamma_q(z)} = \frac{1}{\Gamma(z)}$ where $\Gamma(z)$ is given by [1,4,6,9, 11,12]

$$\frac{1}{\Gamma(z)} = z\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z}, \quad z \in \mathbb{C}.$$

The q-shifted factorials of $a, a_1, \ldots, a_m \in \mathbb{C}$ are given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad (a_1,\ldots,a_m;q)_n = \prod_{k=1}^{m}(a_k;q)_n \tag{2.2}$$

for all integers $n \in \mathbb{Z}$ and $m \in \mathbb{N}$.

Given two sets of complex numbers $\{a_1,\ldots,a_r\}$ and $\{b_1,\ldots,b_s\}$, the basic hypergeometric series ${}_r\phi_s$ is formally defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\,\middle|\, q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k (zq^{-\ell})^k q^{\ell k^2}}{(q,b_1,\ldots,b_s;q)_k (-1)^{k(s+1-r)}}, \tag{2.3}$$

where $\ell = \frac{s+1-r}{2}$, and it defines a confluent basic hypergeometric series if $\ell > 0$.

The four Jacobi theta functions are defined by [11,12]

$$\theta_1(v|\tau) = -i\sum_{k=-\infty}^{\infty}(-1)^k q^{(k+1/2)^2} e^{(2k+1)\pi i v} = 2q^{1/4}\sin\pi v\bigl(q^2, q^2e^{2\pi i v}, q^2 e^{-2\pi i v}; q^2\bigr)_\infty,$$

$$\theta_2(v|\tau) = \sum_{k=-\infty}^{\infty} q^{(k+1/2)^2} e^{(2k+1)\pi i v} = 2q^{1/4}\cos\pi v\bigl(q^2, -q^2e^{2\pi i v}, -q^2 e^{-2\pi i v}; q^2\bigr)_\infty,$$

$$\theta_3(v|\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2k\pi i v} = \bigl(q^2, -qe^{2\pi i v}, -qe^{-2\pi i v}; q^2\bigr)_\infty,$$

$$\theta_4(v|\tau) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2k\pi i v} = \bigl(q^2, qe^{2\pi i v}, qe^{-2\pi i v}; q^2\bigr)_\infty,$$

where $q = e^{\pi i\tau}$ for $\Im(\tau) > 0$. Then,

$$\theta_1\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_1(v|\tau), \qquad \theta_2\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_4(v|\tau),$$
$$\theta_3\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_3(v|\tau), \qquad \theta_4\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_2(v|\tau).$$

For our convenience, we also use the following notations

$$\theta_\lambda(z;q) = \theta_\lambda(v|\tau), \quad z = e^{2\pi i v},\ q = e^{\pi i\tau},\ \lambda = 1,2,3,4.$$

Let x be a real number, we write $x = \lfloor x\rfloor + \{x\}$, where the fractional part of x is $\{x\} \in [0,1)$ and $\lfloor x\rfloor \in \mathbb{Z}$ is the greatest integer less than or equal to x. The function

$$\chi(n) = \begin{cases} 1, & 2\nmid n, \\ 0, & 2|n \end{cases}$$

satisfies

$$\chi(n) = 2\left\{\frac{n}{2}\right\} = n - 2\left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor$$

and

$$\left\lfloor\frac{n+1}{2}\right\rfloor = \frac{n+\chi(n)}{2}, \qquad \left\lfloor\frac{n}{2}\right\rfloor = \frac{n-\chi(n)}{2}.$$

3. Main results

Definition 1. A sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers is said to be admissible if

(1) $\lim_{n\to\infty}\frac{\lambda_n}{\log n} = \infty$,
(2) $\lim_{n\to\infty}\frac{\lambda_n}{\sqrt{n}} = 0$.

Clearly, $\lambda_n = n^\beta\log^\gamma n$, $0 < \beta < \frac{1}{2}$, $\gamma \geqslant 0$ and $\lambda_n = \log^\gamma n$, $\gamma > 1$ are admissible.

Given non-negative integers r, s, t and a positive number ℓ, we define [13]

$$g(a_1,\ldots,a_r;b_1,\ldots,b_s;q;\ell;z) = \sum_{k=0}^\infty \frac{(q^{k+1},b_1q^k,\ldots,b_sq^k;q)_\infty q^{\ell k^2}(-z)^k}{(a_1q^k,\ldots,a_rq^k;q)_\infty}, \tag{3.1}$$

$$h_n(a_1,\ldots,a_r;b_1,\ldots,b_s;c_1,\ldots,c_t;q;\ell;z) = \sum_{k=0}^n \frac{(q^{k+1},b_1q^k,\ldots,b_sq^k;q)_\infty q^{\ell k^2}(-z)^k}{(a_1q^k,\ldots,a_rq^k;q)_\infty}\frac{(q,c_1,\ldots,c_t;q)_n}{(q,c_1,\ldots,c_t;q)_{n-k}}, \tag{3.2}$$

where

$$0 \leqslant a_1, \dots, a_r, b_1, \dots, b_s, c_1, \dots, c_t < 1. \tag{3.3}$$

In order to simplify the type setting we let

$$\begin{aligned} g(z;q) &= g(a_1, \dots, a_r; b_1, \dots, b_s; q; \ell; z), \\ h(z;q) &= h_n(a_1, \dots, a_r; b_1, \dots, b_s; c_1, \dots, c_t; q; \ell; z) \end{aligned}$$

in the following theorem:

Theorem 2. *Given an admissible sequence λ_n, assume that $z = e^{2\pi v}$, $q = e^{-\pi\lambda_n^{-1}}$, $\ell > 0$, $v \in \mathbb{R}$ and $a_j = q^{\alpha_j}$, $b_k = q^{\beta_k}$, $\alpha_j, \beta_k > 0$ for $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant s$. Then*

$$\begin{aligned} g(-q^{-4n\ell}z;q) &= \sqrt{\frac{\lambda_n}{\ell}} \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(e^{-\ell^{-1}\pi\lambda_n}\right)\right\}, \\ h(-zq^{-n\ell};q) &= \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}\right\}\sqrt{\frac{\lambda_n}{\ell}}\left\{1 + \mathcal{O}\left(e^{-\ell^{-1}\pi\lambda_n}\right)\right\}, \\ g(q^{-4n\ell}z;q) &= 2\sqrt{\frac{\lambda_n}{\ell}} \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2 - \frac{\pi\lambda_n}{4\ell}\right\}\left\{\cos\frac{\pi\lambda_n v}{\ell} + \mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\}, \\ h(zq^{-n\ell};q) &= \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi\lambda_n}{4\ell}\right\} \\ &\quad \times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n}{\ell}\left(v + \frac{\ell(n-\chi(n))}{2\lambda_n}\right) + \mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Ramanujan's entire function

$$A_q(z) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-z)^n}{(q;q)_n} = \frac{g(-;-;q;1;z)}{(q;q)_\infty} \tag{3.4}$$

we have:

Corollary 3. *Given an admissible sequence λ_n, assume that $z = e^{2\pi v}$, $q = e^{-\pi\lambda_n^{-1}}$ and $v \in \mathbb{R}$. Then*

$$A_q(-q^{-4n}z) = \frac{1}{\sqrt{2}} \exp\left\{\pi\lambda_n\left(v + \frac{2n}{\lambda_n}\right)^2 + \frac{\pi\lambda_n}{6} - \frac{\pi}{24\lambda_n}\right\}\left\{1 + \mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\}$$

and

$$A_q(q^{-4n}z) = \sqrt{2} \exp\left\{\pi\lambda_n\left(v + \frac{2n}{\lambda_n}\right)^2 - \frac{\pi\lambda_n}{12} - \frac{\pi}{24\lambda_n}\right\}\left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Jackson's q-Bessel function

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k\nu}(-1)^k}{(q,q^{\nu+1};q)_k} \left(\frac{z}{2}\right)^{2k+\nu}$$
$$= \frac{g(-;q^{\nu+1};q;1;z^2q^\nu/4)}{(q;q)_\infty^2(2/z)^\nu}, \quad \nu > -1 \tag{3.5}$$

we have:

Corollary 4. *For an admissible sequence λ_n, assume that $z = e^{2\pi\nu}$, $q = e^{-\pi\lambda_n^{-1}}$, $\nu \in \mathbb{R}$ and $\nu > -1$. Then*

$$J_\nu^{(2)}\big(2i\sqrt{zq^{-\nu}}q^{-2n};q\big) = \frac{\exp(\frac{\pi\lambda_n}{3} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n} + \frac{\nu\pi i}{2})}{2\sqrt{\lambda_n}}$$
$$\times \exp\left\{\pi\lambda_n\left(\nu + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\big(e^{-\pi\lambda_n}\big)\right\}$$

and

$$J_\nu^{(2)}\big(2\sqrt{zq^{-\nu}}q^{-2n};q\big) = \frac{\exp(\frac{\pi\lambda_n}{12} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n})}{\sqrt{\lambda_n}}$$
$$\times \exp\left\{\pi\lambda_n\left(\nu + \frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{\cos\pi\lambda_n\nu + \mathcal{O}\big(e^{-2\pi\lambda_n}\big)\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for ν in any compact subset of $\mathbb{R}$.

For the confluent basic hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\middle| q, z(-1)^{s-r}\right) = \frac{(a_1,\dots,a_r;q)_\infty g(a_1,\dots,a_r;b_1,\dots b_s;q;\ell;zq^{-\ell})}{(q,b_1,\dots,b_s;q)_\infty} \tag{3.6}$$

we have:

Corollary 5. *Given an admissible sequence λ_n, assume that $z = e^{2\pi\nu}$, $q = e^{-\pi\lambda_n^{-1}}$, $\nu \in \mathbb{R}$, and $\alpha_j, \beta_k > 0$, $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant s$. Let $\ell = \frac{s+1-r}{2} > 0$, and $\rho = \sum_{j=1}^r \alpha_j - \sum_{k=1}^s \beta_k - 1$. Then*

$${}_r\phi_s\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s} \end{matrix}\middle| q, (-1)^{s+1-r}zq^{-\ell(4n-1)}\right)$$
$$= \frac{\prod_{k=1}^s \Gamma(\beta_k)}{\prod_{j=1}^r \Gamma(\alpha_j)} \frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell}2^\ell\pi^{\rho+2\ell}} \left\{\exp\frac{\pi\lambda_n}{\ell}\left(\nu + \frac{2n\ell}{\lambda_n}\right)^2 + \ell\pi\lambda_n/3\right\}\left\{1 + \mathcal{O}\big(\lambda_n^{-1}\big)\right\}$$

and

$$
\begin{aligned}
&{}_r\phi_s\left(\begin{matrix} q^{\alpha_1}, \dots, q^{\alpha_r} \\ q^{\beta_1}, \dots, q^{\beta_s} \end{matrix} \,\middle|\, q, (-1)^{s-r} z q^{-\ell(4n-1)}\right) \\
&\quad = \frac{\prod_{k=1}^{s} \Gamma(\beta_k)}{\prod_{j=1}^{r} \Gamma(\alpha_j)} \frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell} 2^{\ell-1} \pi^{\rho+2\ell}} \left\{ \exp \frac{\pi \lambda_n}{\ell} \left(v + \frac{2n\ell}{\lambda_n} \right)^2 + \frac{\ell \pi \lambda_n}{3} - \frac{\pi \lambda_n}{4\ell} \right\} \\
&\qquad \times \left\{ \cos \frac{\pi \lambda_n v}{\ell} + \mathcal{O}\big(\lambda_n^{-1}\big) \right\}
\end{aligned}
$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the q^{-1}-Hermite polynomials

$$
h_n(\sinh \xi | q) = \sum_{k=0}^{n} \frac{(q;q)_n q^{k(k-n)} (-1)^k e^{(n-2k)\xi}}{(q;q)_k (q;q)_{n-k}} = \frac{h_n(-;-;-;q;1;e^{-2\xi} q^{-n})}{e^{-n\xi} (q;q)_\infty} \tag{3.7}
$$

we have:

Corollary 6. *Given an admissible sequence λ_n, for any $v \in \mathbb{R}$. Then*

$$
\begin{aligned}
h_n\left(\sinh \pi \left(v + \frac{i}{2} \right) \middle| q \right) &= \frac{\exp\{\frac{\pi n^2}{4\lambda_n} + \frac{\pi \lambda_n}{6} - \frac{\pi(1+12\chi(n))}{24\lambda_n}\}}{(-i)^n \sqrt{2}} \left\{ \exp\left[\pi \lambda_n \left(v - \frac{\chi(n)}{2\lambda_n} \right)^2 \right] \right\} \\
&\quad \times \left\{ 1 + \mathcal{O}\big(e^{-\pi \lambda_n}\big) \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
h_n(\sinh \pi v | q) &= (-1)^n \sqrt{2} \exp\left\{ \frac{n^2 \pi}{4\lambda_n} - \frac{(1+12\chi(n))\pi}{24\lambda_n} - \frac{\pi \lambda_n}{12} \right\} \left\{ \exp\left[\pi \lambda_n \left(v - \frac{\chi(n)}{2\lambda_n} \right)^2 \right] \right\} \\
&\quad \times \left\{ \cos \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right) + \mathcal{O}\big(e^{-2\pi \lambda_n}\big) \right\}
\end{aligned}
$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Stieltjes–Wigert polynomials

$$
S_n(x;q) = \sum_{k=0}^{n} \frac{q^{k^2} (-x)^k}{(q;q)_k (q;q)_{n-k}} = \frac{h_n(-;-;-;q;1;x)}{(q;q)_n (q;q)_\infty} \tag{3.8}
$$

we have:

Corollary 7. *Given an admissible sequence λ_n, assume that $z = e^{2\pi v}$, $q = e^{-\pi \lambda_n^{-1}}$ and $v \in \mathbb{R}$. Then*

$$
\begin{aligned}
S_n\big(-zq^{-n}; q\big) &= \frac{\exp\{\frac{\pi \lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\}}{2\sqrt{\lambda_n}} \left\{ \exp \pi \lambda_n \left(v + \frac{n - \chi(n)}{2\lambda_n} \right)^2 \right\} \\
&\quad \times \left\{ 1 + \mathcal{O}\big(e^{-\pi \lambda_n}\big) \right\}
\end{aligned}
$$

and

$$S_n\big(zq^{-n};q\big)=\frac{\exp\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}$$
$$\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\big(e^{-2\pi\lambda_n}\big)\right\}$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q)=\sum_{k=0}^{n}\frac{q^{k^2+\alpha k}(-x)^k(q^{\alpha+1};q)_n}{(q;q)_k(q,q^{\alpha+1};q)_{n-k}}=\frac{h_n(-;-;q^{\alpha+1};q;1;xq^{\alpha})}{(q;q)_n(q;q)_\infty} \tag{3.9}$$

we have the following:

Corollary 8. *Given an admissible sequence λ_n, assume that $z=e^{2\pi v}$, $q=e^{-\pi\lambda_n^{-1}}$, $v\in\mathbb{R}$ and $\alpha>-1$. Then*

$$L_n^{(\alpha)}\big(-zq^{-\alpha-n};q\big)=\frac{\exp\{\frac{\pi\lambda_n}{3}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{2\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}$$
$$\times\big\{1+\mathcal{O}\big(e^{-\pi\lambda_n}\big)\big\}$$

and

$$L_n^{(\alpha)}\big(zq^{-\alpha-n};q\big)=\frac{\exp\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}$$
$$\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\big(e^{-2\pi\lambda_n}\big)\right\}$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

4. Proofs

The following lemma is from [13]:

Lemma 9. *Given $a\in\mathbb{C}$ with $0<\frac{|a|q^n}{1-q}<\frac{1}{2}$ for some $n\in\mathbb{N}$. Then*

$$\frac{(a;q)_\infty}{(a;q)_n}=\big(aq^n;q\big)_\infty=1+r_1(a;n)$$

with $|r_1(a;n)|\leqslant\frac{2|a|q^n}{1-q}$ and

$$\frac{(a;q)_n}{(a;q)_\infty}=\frac{1}{(aq^n;q)_\infty}=1+r_2(a;n)$$

with $|r_2(a;n)|\leqslant\frac{2|a|q^n}{1-q}$.

We also need the following lemma:

Lemma 10. *Given a sequence of positive numbers* $\{\lambda_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty}\lambda_n=\infty$. *Let* $q=e^{-\pi\lambda_n^{-1}}$, $x>0$. *Then*

$$(q^x;q)_\infty=\frac{\sqrt{2}\pi^{1-x}\lambda_n^{x-1/2}}{\Gamma(x)\exp(\pi\lambda_n/6)}\bigl\{1+\mathcal{O}\bigl(\lambda_n^{-1}\bigr)\bigr\}$$

as $n\to\infty$.

Proof. For $x\neq 0,-1,-2,\ldots$ and let $q=e^{-t}$, the McIntosh asymptotic formula [10] is

$$\log\bigl(q^x;q\bigr)_\infty=-\frac{\pi^2}{6t}+\left(\frac{1}{2}-x\right)\log t+\frac{\log(2\pi)}{2}-\log\Gamma(x)$$
$$+\sum_{k=1}^{p}\frac{B_kB_{k+1}(x)}{k(k+1)!}t^k+\mathcal{O}\bigl(t^{p+1}\bigr)$$

for any positive integer p as $t\to 0^+$, where B_k and $B_k(x)$ are the kth Bernoulli number and the kth Bernoulli polynomial respectively. Take the main term in the McIntosh asymptotic formula with $t=\frac{\pi}{\lambda_n}$ and Lemma 10 follows. $\square$

Take $\lambda=0$, $\tau=2$, $m=2n$ in Theorem 2.2 of [15] we get the following result:

Lemma 11. *Assume that* $z\in\mathbb{C}\backslash\{0\}$, $\ell>0$ *and* (3.3). *Then*

$$g\bigl(q^{-4n\ell}z;q\bigr)=z^{2n}q^{-4n^2\ell}\bigl\{\theta_4\bigl(z^{-1};q^{\ell}\bigr)+r_g(n|1)\bigr\}$$

and

$$|r_g(n|1)|\leqslant\frac{2^{s+r+3}\theta_3(|z|^{-1};q^{\ell})}{(a_1,\ldots,a_r;q)_\infty}\left\{\frac{q^{n+1}}{1-q}+\frac{q^{\ell n^2}}{|z|^n}\right\}$$

for n sufficiently large. In particular,

$${}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\middle|q,(-1)^{s-r}zq^{-4n\ell}\right)=\frac{(a_1,\ldots,a_r;q)_\infty z^{2n}\{\theta_4(z^{-1}q^{\ell};q^{\ell})+r_\phi(n|1)\}}{(q,b_1,\ldots,b_s;q)_\infty q^{2\ell n(2n+1)}}$$

and

$$\bigl|r_\phi(n|1)\bigr|\leqslant\frac{2^{s+r+3}\theta_3(|z|^{-1}q^{\ell};q^{\ell})}{(a_1,\ldots,a_r;q)_\infty}\left\{\frac{q^{n+1}}{1-q}+\frac{q^{\ell n^2+\ell n}}{|z|^n}\right\}$$

for n sufficiently large, where $\ell=\frac{s+1-r}{2}>0$.

Similarly, if we take $\lambda = 0$ and $\tau = \frac{1}{2}$ in Theorem 2.4 of [15] we get:

Lemma 12. *Assume that $z \in \mathbb{C}\backslash\{0\}$, $\ell > 0$ and (3.3). Then*

$$h_n\bigl(zq^{-n\ell}; q\bigr) = (-z)^{\lfloor n/2\rfloor} q^{-\ell[n^2-\chi(n)]/4}\bigl\{\theta_4\bigl(z^{-1}; q^\ell\bigr) + r_h(n|1)\bigr\}$$

and

$$\bigl|r_h(n|1)\bigr| \leqslant \frac{2^{s+r+2t+5}\theta_3(|z|^{-1}; q^\ell)}{(a_1, \ldots, a_r; q)_\infty}\left\{\frac{q^{\lfloor n/4\rfloor+1}}{1-q} + |z|^{\lfloor n/4\rfloor} q^{\ell\lfloor n/4\rfloor^2} + \frac{q^{\ell\lfloor n/4\rfloor^2}}{|z|^{\lfloor n/4\rfloor}}\right\}$$

for n sufficiently large.

4.1. Proof for Theorem 2

From the formulas of θ_3 we obtain

$$\begin{aligned}\theta_3\bigl(e^{-2\pi v}; e^{-\pi\ell\lambda_n^{-1}}\bigr) &= \theta_3\bigl(vi|\ell\lambda_n^{-1}i\bigr) = \sqrt{\frac{\lambda_n}{\ell}} e^{\pi\ell^{-1}\lambda_n v^2}\theta_3\left(\frac{\lambda_n v}{\ell}\middle|\frac{\lambda_n i}{\ell}\right) \\ &= \sqrt{\frac{\lambda_n}{\ell}} e^{\pi\ell^{-1}\lambda_n v^2}\bigl\{1 + \mathcal{O}\bigl(e^{-\ell^{-1}\pi\lambda_n}\bigr)\bigr\} \qquad (4.1)\end{aligned}$$

as $n \to \infty$, uniformly for all $v \in \mathbb{R}$. Clearly we have

$$\frac{q^{n+1}}{1-q} + q^{\ell n^2} e^{-2n\pi v} = \mathcal{O}\bigl(\lambda_n e^{-\pi n\lambda_n^{-1}}\bigr)$$

as $n \to \infty$, uniformly for v in any compact subset of $\mathbb{R}$. From Lemma 10 we have

$$\bigl(q^{\alpha_1}, \ldots, q^{\alpha_r}; q\bigr)_\infty = \frac{2^{r/2}\pi^{r-\sum_{j=1}^r \alpha_j}\{1 + \mathcal{O}(\lambda_n^{-1})\}}{e^{r\pi\lambda_n/6}\lambda_n^{r/2-\sum_{j=1}^r \alpha_j}\prod_{j=1}^r \Gamma(\alpha_j)} \qquad (4.2)$$

as $n \to \infty$. Since λ_n is admissible, then

$$g\bigl(-q^{-4n\ell}z; q\bigr) = \sqrt{\frac{\lambda_n}{\ell}} \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2\right\}\bigl\{1 + \mathcal{O}\bigl(e^{-\ell^{-1}\pi\lambda_n}\bigr)\bigr\}$$

as $n \to \infty$, uniformly for v in any compact subset of $\mathbb{R}$.

Since

$$\begin{aligned}\theta_4\bigl(z^{-1}; q^\ell\bigr) &= \theta_4\bigl(e^{-2\pi v}; e^{-\ell\pi\lambda_n^{-1}}\bigr) = \theta_4\left(vi\middle|\frac{\ell i}{\lambda_n}\right) \\ &= \sqrt{\frac{\lambda_n}{\ell}} e^{\pi\ell^{-1}\lambda_n v^2}\theta_2\left(\frac{\lambda_n v}{\ell}\middle|\frac{i\lambda_n}{\ell}\right) \\ &= 2\sqrt{\frac{\lambda_n}{\ell}} \exp\left(\frac{\pi\lambda_n v^2}{\ell} - \frac{\pi\lambda_n}{4\ell}\right)\cos\frac{\pi\lambda_n v}{\ell}\bigl\{1 + \mathcal{O}\bigl(e^{-2\pi\ell^{-1}\lambda_n}\bigr)\bigr\} \qquad (4.3)\end{aligned}$$

as $n \to \infty$, uniformly in $v \in \mathbb{R}$. Thus,

$$g\left(q^{-4n\ell}z;q\right) = 2\sqrt{\frac{\lambda_n}{\ell}}\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2 - \frac{\pi\lambda_n}{4\ell}\right\}\left\{\cos\frac{\pi\lambda_n v}{\ell} + \mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, uniformly on any compact subset of $\mathbb{R}$.

Clearly,

$$\frac{q^{\lfloor n/4\rfloor+1}}{1-q} + |z|^{\lfloor n/4\rfloor}q^{\ell\lfloor n/4\rfloor^2} + \frac{q^{\ell\lfloor n/4\rfloor^2}}{|z|^{\lfloor n/4\rfloor}} = \mathcal{O}\left(e^{-\pi n/(4\lambda_n)}\right) \tag{4.4}$$

as $n \to \infty$, uniformly on any compact subset of $\mathbb{R}$. From Eqs. (4.1), (4.2) and (4.4) we get

$$h\left(-zq^{-n\ell};q\right) = \sqrt{\frac{\lambda_n}{\ell}}\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}\right\}\left\{1+\mathcal{O}\left(e^{-\ell^{-1}\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, uniformly on any compact subset of $\mathbb{R}$. Using Eqs. (4.1), (4.2), (4.3) and (4.4) we obtain

$$\begin{aligned} h\left(zq^{-n\ell};q\right) &= \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2 + \frac{\ell\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi\lambda_n}{4\ell}\right\} \\ &\quad \times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right) + \mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n \to \infty$, uniformly on any compact subset of $\mathbb{R}$.

4.2. *Proof for Corollary 3*

By Lemma 10 and Theorem 2 we have

$$\begin{aligned} A_q\left(-q^{-4n}z\right) &= \frac{g(-;-;q;1;-q^{-4n}z)}{(q;q)_\infty} \\ &= \frac{1}{\sqrt{2}}\exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2 + \frac{\pi\lambda_n}{6} - \frac{\pi}{24\lambda_n}\right\}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\} \end{aligned}$$

and

$$\begin{aligned} A_q\left(q^{-4n}z\right) &= \frac{g(-;-;q;1;q^{-4n}z)}{(q;q)_\infty} \\ &= \sqrt{2}\exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2 - \frac{\pi\lambda_n}{12} - \frac{\pi}{24\lambda_n}\right\}\left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n \to \infty$, uniformly on any compact subset of $\mathbb{R}$.

4.3. Proof for Corollary 4

Apply Lemma 10 and Theorem 2 to get

$$
\begin{aligned}
J_\nu^{(2)}\big(2i\sqrt{zq^{-\nu}}q^{-2n};q\big) &= \frac{g(-;q^{\nu+1};q;1;-zq^{-4n})}{(q;q)_\infty^2(i\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}} \\
&= \frac{\exp(\frac{\pi\lambda_n}{3}-\frac{\pi}{12\lambda_n}+\frac{\nu^2\pi}{4\lambda_n}+\frac{\nu\pi i}{2})}{2\sqrt{\lambda_n}}\exp\left\{\pi\lambda_n\left(\nu+\frac{4n+\nu}{2\lambda_n}\right)^2\right\}\left\{1+\mathcal{O}\big(e^{-\pi\lambda_n}\big)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
J_\nu^{(2)}\big(2\sqrt{zq^{-\nu}}q^{-2n};q\big) &= \frac{g(-;q^{\nu+1};q;1;zq^{-4n})}{(q;q)_\infty^2(\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}} \\
&= \frac{\exp(\frac{\pi\lambda_n}{12}-\frac{\pi}{12\lambda_n}+\frac{\nu^2\pi}{4\lambda_n})}{\sqrt{\lambda_n}}\exp\left\{\pi\lambda_n\left(\nu+\frac{4n+\nu}{2\lambda_n}\right)^2\right\} \\
&\quad\times\left\{\cos\pi\lambda_n\nu+\mathcal{O}\big(e^{-2\pi\lambda_n}\big)\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.4. Proof for Corollary 5

Apply Lemma 10, Lemma 10 and Theorem 2 to get

$$
\begin{aligned}
&{}_r\phi_s\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s}\end{matrix}\,\middle|\, q,(-1)^{s+1-r}zq^{-\ell(4n-1)}\right) \\
&\quad= \frac{(q^{\alpha_1},\dots,q^{\alpha_r};q)_\infty g(-q^{-4n\ell}z;q)}{(q,q^{\beta_1},\dots,q^{\beta_s};q)_\infty} \\
&\quad= \frac{\prod_{k=1}^s\Gamma(\beta_k)}{\prod_{j=1}^r\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{2^\ell\pi^{\rho+2\ell}\sqrt{\ell}}\left\{\exp\frac{\pi\lambda_n}{\ell}\left(\nu+\frac{2n\ell}{\lambda_n}\right)^2+\ell\pi\lambda_n/3\right\}\left\{1+\mathcal{O}\big(\lambda_n^{-1}\big)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_s\phi_r\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s}\end{matrix}\,\middle|\, q,(-1)^{s-r}zq^{-\ell(4n-1)}\right) \\
&\quad= \frac{(q^{\alpha_1},\dots,q^{\alpha_r};q)_\infty g(q^{-4n\ell}z;q)}{(q,q^{\beta_1},\dots,q^{\beta_s};q)_\infty} \\
&\quad= \frac{\prod_{k=1}^s\Gamma(\beta_k)}{\prod_{j=1}^r\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell}2^{\ell-1}\pi^{\rho+2\ell}}\left\{\exp\frac{\pi\lambda_n}{\ell}\left(\nu+\frac{2n\ell}{\lambda_n}\right)^2+\frac{\ell\pi\lambda_n}{3}-\frac{\pi\lambda_n}{4\ell}\right\}\left\{\cos\frac{\pi\lambda_n\nu}{\ell}+\mathcal{O}\big(\lambda_n^{-1}\big)\right\}.
\end{aligned}
$$

4.5. Proof for Corollary 6

For $v \in \mathbb{R}$, Lemma 10 and Theorem 2 imply

$$\begin{aligned}
&h_n\left(\sinh \pi\left(v+\frac{i}{2}\right)\middle| q\right) \\
&\quad = \frac{h(-;-;-;q;1;-e^{2\pi v}q^{-n})}{(-i)^n e^{n\pi v}(q;q)_\infty} \\
&\quad = \frac{\exp\{\frac{\pi n^2}{4\lambda_n}+\frac{\pi\lambda_n}{6}-\frac{\pi(1+12\chi(n))}{24\lambda_n}\}}{(-i)^n\sqrt{2}}\left\{\exp\left[\pi\lambda_n\left(v-\frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
h_n(\sinh\pi v|q) &= \frac{h(-;-;-;q;1;e^{2\pi v}q^{-n})}{(-1)^n e^{n\pi v}(q;q)_\infty} \\
&= (-1)^n\sqrt{2}\exp\left\{\frac{n^2\pi}{4\lambda_n}-\frac{(1+12\chi(n))\pi}{24\lambda_n}-\frac{\pi\lambda_n}{12}\right\}\left\{\exp\left[\pi\lambda_n\left(v-\frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\} \\
&\quad\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}
\end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.6. Proof for Corollary 7

From Lemma 9 and Lemma 10 we obtain

$$\frac{1}{(q;q)_n(q;q)_\infty} = \frac{1}{(q;q)_\infty^2}\frac{(q;q)_\infty}{(q;q)_n} = \frac{\exp\{\frac{\pi\lambda_n}{3}-\frac{\pi}{12\lambda_n}\}}{2\lambda_n}\left\{1+\mathcal{O}\left(e^{-4\pi\lambda_n}\right)\right\} \tag{4.5}$$

as $n\to\infty$. Hence, Theorem 2 implies

$$\begin{aligned}
S_n\left(-zq^{-n};q\right) &= \frac{h(-;-;-;q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\{\frac{\pi\lambda_n}{3}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{2\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
S_n\left(zq^{-n};q\right) &= \frac{h(-;-;-;q;1;zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\} \\
&\quad\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}
\end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.7. Proof for Corollary 8

From (4.5) and Theorem 2 we obtain

$$
\begin{aligned}
L_n^{(\alpha)}\bigl(-zq^{-\alpha-n};q\bigr) &= \frac{h(-;-;q^{\alpha+1};q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\{\frac{\pi\lambda_n}{3}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{2\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(\nu+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\bigl\{1+\mathcal{O}\bigl(e^{-\pi\lambda_n}\bigr)\bigr\}
\end{aligned}
$$

and

$$
\begin{aligned}
L_n^{(\alpha)}\bigl(zq^{-\alpha-n};q\bigr) &= \frac{h(-;-;q^{\alpha+1};q;1;zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\}}{\sqrt{\lambda_n}}\left\{\exp\pi\lambda_n\left(\nu+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\} \\
&\quad\times\left\{\cos\pi\lambda_n\left(\nu+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\bigl(e^{-2\pi\lambda_n}\bigr)\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

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References

[1] G.E. Andrews, R.A. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[2] Y. Chen, M.E.H. Ismail, K.A. Muttalib, Asymptotics of basic Bessel functions and q-Laguerre polynomials, J. Comput. Appl. Math. 54 (1995) 263–273.
[3] A.B. Olde Daalhuis, Asymptotic expansions for q-gamma, q-exponential, and q-Bessel functions, J. Math. Anal. Appl. 186 (1994) 896–913.
[4] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 2004.
[5] M.E.H. Ismail, Asymptotics of q-orthogonal polynomials and a q-Airy function, Int. Math. Res. Not. IMRN (2005) 1063–1088.
[6] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge University Press, Cambridge, 2005.
[7] M.E.H. Ismail, R. Zhang, Chaotic and periodic asymptotics for q-orthogonal polynomials, Int. Math. Res. Not. IMRN (2006) 1–34.
[8] M.E.H. Ismail, R. Zhang, Scaled asymptotics for q-orthogonal polynomials, C. R. Math. Acad. Sci. Paris 344 (2007) 71–75.
[9] R. Koekoek, P.A. Lesky, R. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer-Verlag, Berlin, 2010.
[10] R.J. McIntosh, Some asymptotic formulae for q-shifted factorials, Ramanujan J. 3 (1999) 205–214.
[11] H. Rademacher, Topics in Analytic Number Theory, Springer-Verlag, New York, 1973.
[12] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge, 1927.
[13] R. Zhang, Plancherel–Rotach asymptotics for certain basic hypergeometric series, Adv. Math. 217 (2008) 1588–1613.
[14] R. Zhang, Plancherel–Rotach asymptotics for some q-orthogonal polynomials with complex scalings, Adv. Math. 218 (2008) 1051–1080.
[15] R. Zhang, Scaled asymptotics for some q-series, Q. J. Math. 59 (2008) 389–407.